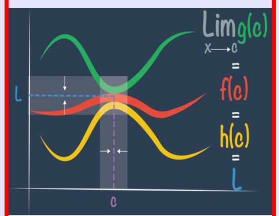
# Calculus I

# Lecture 26




Feb 19-8:47 AM

Evaluate $\int_0^1 x^3\sqrt{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\int_0^1 x^3\sqrt{x^2+1}\,dx = \int_0^1 x\,x^2\sqrt{x^2+1}\,dx$$

$$= \int_1^2 (u-1)\sqrt{u}\,\frac{du}{2} = \frac{1}{2}\int_1^2 (u\sqrt{u}-\sqrt{u})\,du$$

$$= \frac{1}{2}\int_1^2 (u^{3/2}-u^{1/2})\,du$$

$$= \frac{1}{2}\left[\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right]\Big|_1^2$$

$$= \left(\frac{u^{5/2}}{5} - \frac{u^{3/2}}{3}\right)\Big|_1^2$$

$$= \left(\frac{u^2\sqrt{u}}{5} - \frac{u\sqrt{u}}{3}\right)\Big|_1^2$$

$$= \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= \frac{12\sqrt{2}-10\sqrt{2}}{15} - \frac{3-5}{15} = \frac{2\sqrt{2}}{15} + \frac{2}{15} = \boxed{\frac{2\sqrt{2}+2}{15}}$$

Jul 30-8:02 AM

Find $\int x^2 \sqrt{x+2}\, dx$

$u = \sqrt{x+2}$ $\quad x = u^2 - 2$

$u^2 = x + 2$

$2u\,du = dx$

$$= \int (u^2-2)^2 u \cdot 2u\,du$$

$$= 2\int (u^4 - 4u^2 + 4)u^2\,du =$$

$$= 2\int (u^6 - 4u^4 + 4u^2)\,du = 2\left[\frac{u^7}{7} - \frac{4u^5}{5} + \frac{4u^3}{3}\right] + C$$

$$= 2\left[\frac{(\sqrt{x+2})^7}{7} - \frac{4(\sqrt{x+2})^5}{5} + \frac{4(\sqrt{x+2})^3}{3}\right] + C$$

Jul 30-8:11 AM

Evaluate $\int_{-\pi}^{\pi} \underbrace{x^4 \sin x}_{\text{odd function}}\, dx$

$= \boxed{0}$

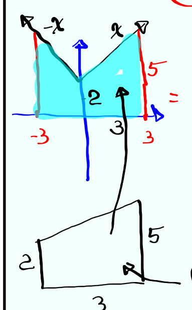
$\sin x$ is an odd function.

$x^4 \sin x$ is also odd function.

Evaluate $\int_{10}^{10} \sqrt{\cos x}\, dx = 0$

Evaluate $\int_{-3}^{3} (\underbrace{|x|+2}_{\text{even}})\,dx$

$|x|$ is an even function




$$= 2\int_0^3 (|x|+2)\,dx = 2\int_0^3 (x+2)\,dx$$

$$= 2\left[\frac{x^2}{2} + 2x\right]\Big|_0^3 = 2\left(\frac{9}{2}+6\right) = 9 + 12$$

$= \boxed{21}$

$A = \frac{3(2+5)}{2} = \frac{21}{2}$ $\qquad 2\left(\frac{21}{2}\right) = 21$

Jul 30-8:18 AM

Evaluate $\int_{-5}^{5} \sqrt{25-x^2}\,dx$

1) Subs. → will not work

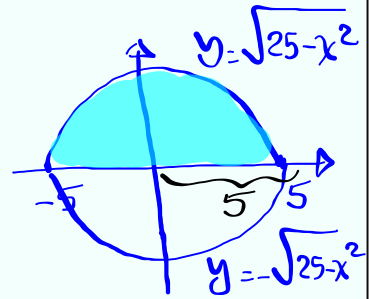
2) Subs. in Calc. II → will work

$= \frac{1}{2}$ Area of circle with radius 5

Let $y = \sqrt{25-x^2}$

$y^2 = 25 - x^2$

$x^2 + y^2 = 25$




$= \frac{1}{2} \cdot \pi (5)^2 = \boxed{\frac{25\pi}{2}}$

Jul 30-8:28 AM

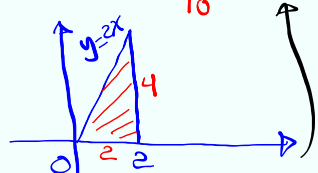
Use Reimann Sum to evaluate $\int_0^2 2x\,dx$

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$

$= x^2 \Big|_0^2 = \boxed{4}$

$\Delta x = \frac{b-a}{n}$ $\quad x_i = a + i\Delta x$

$a=0 \quad b=2 \quad f(x)=2x$




$A = \frac{bh}{2} = \frac{2\cdot 4}{2} = \boxed{4}$

$\Delta x = \frac{2-0}{n} = \frac{2}{n}$ $\quad x_i = 0 + i\cdot\frac{2}{n} = \frac{2i}{n}$

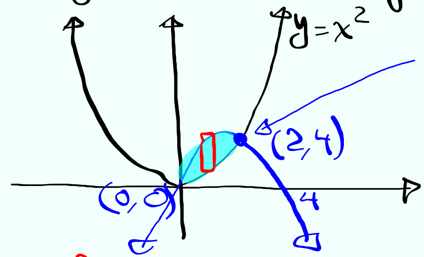
$f(x_i) = 2x_i = 2\cdot\frac{2i}{n} = \frac{4i}{n}$

$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \frac{4i}{n}\cdot\frac{2}{n} = \sum_{i=1}^{n} \frac{8}{n^2} i = \frac{8}{n^2}\sum_{i=1}^{n} i = \frac{8}{n^2}\cdot\frac{n(n+1)}{2}$

$\int_0^2 2x\,dx = \lim_{n\to\infty} \frac{8n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{8n^2+\cdots}{2n^2} = \frac{8}{2} = \boxed{4}$

Jul 30-8:33 AM

Find the area bounded by $y=x^2$ and $y=4x-x^2$. Drawing Required.




$x^2=4x-x^2$

$x^2-4x+x^2=0$

$2x^2-4x=0$

$2x(x-2)=0$

$x=0 \quad x=2$

$$A=\int_0^2 [\text{Top}-\text{Bottom}]\,dx$$

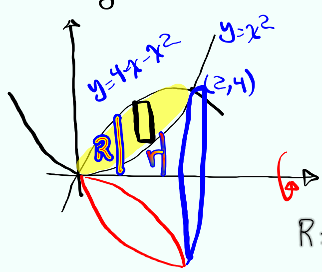
$$=\int_0^2 (4x-x^2-x^2)\,dx=\int_0^2 (4x-2x^2)\,dx=\left(2x^2-\tfrac{2}{3}x^3\right)\Big|_0^2$$

$$=2\cdot 2^2-\tfrac{2}{3}\cdot 2^3$$

$$=8-\tfrac{16}{3}=\boxed{\tfrac{8}{3}}$$

Jul 30-8:42 AM

Rotate the area bounded by $y=x^2$ and $y=4x-x^2$ about the x axis. Find the Volume




Washer

$$V=\int_0^2 \pi\left[R^2-r^2\right]dx$$

$R=4x-x^2 \qquad r=x^2$

$$V=\pi\int_0^2\left[(4x-x^2)^2-(x^2)^2\right]dx$$

$$=\pi\int_0^2\left[16x^2-8x^3\cancel{+x^4}\cancel{-x^4}\right]dx$$

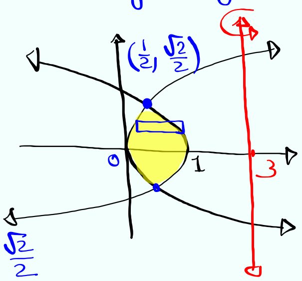
$$=\pi\left[\frac{16x^3}{3}-\frac{8x^4}{4}\right]\Big|_0^2=\pi\left[\frac{16\cdot 8}{3}-2\cdot 16\right]$$

$$=\boxed{\frac{32\pi}{3}}$$

Jul 30-8:42 AM

Rotate the area bounded by $x=y^2$ and $x=1-y^2$ about

1) $x=3$, find Volume.

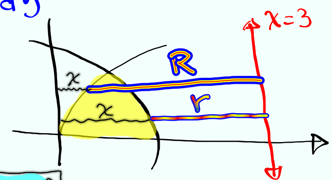



$x = x$

$y^2 = 1-y^2$

$2y^2 = 1$ $\quad y^2 = \frac{1}{2}$ $\quad y = \frac{\sqrt{2}}{2}$

$$V = 2\int_0^{\frac{\sqrt{2}}{2}} \pi (R^2 - r^2)\, dy$$




$x + R = 3$

$y^2 + R = 3$ $\quad \boxed{R = 3-y^2}$

$x + r = 3$ $\quad \boxed{r = 2+y^2}$

$1-y^2+r = 3$

$$V = 2\int_0^{\frac{\sqrt{2}}{2}} \pi \left[(3-y^2)^2 - (2+y^2)^2\right] dy$$

Ans.: $\frac{10\pi\sqrt{2}}{3}$

Jul 30-8:56 AM

Shell Method

use it when Ref. Rect. is Parallel to axis of Revolution.

$$V = \int_a^b 2\pi D \quad dx$$

$$V = \int_c^d 2\pi D H \, dy$$

D is the distance from Ref. Rect. to A.O.R.

H is the length of Ref. Rect.

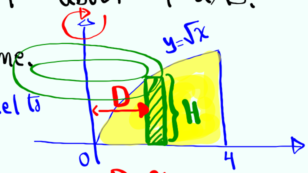
Jul 30-9:09 AM

Rotate the area bounded by $y=\sqrt{x}$, $y=0$, and $x=4$ about Y-axis.

Find the Volume.

Ref. Rect. Parallel to A.O.R.

Shell Method



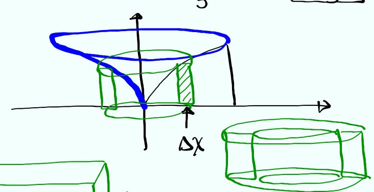

$D=x$

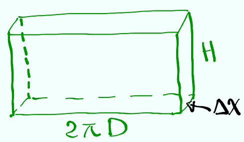
$H=\sqrt{x}$

$$V=\int_0^4 2\pi\, x\sqrt{x}\,dx = 2\pi\int_0^4 x^{3/2}\,dx = 2\pi\cdot\frac{x^{5/2}}{5/2}\Big|_0^4$$

$$=2\pi\cdot\frac{2}{5}\,x^2\sqrt{x}\Big|_0^4 = \frac{4\pi}{5}\left[4^2\sqrt{4}-0\right]$$

$$=\frac{4\pi}{5}\cdot 32 = \boxed{\frac{128\pi}{5}}$$




$V = LWH$

$= 2\pi D H \Delta x$



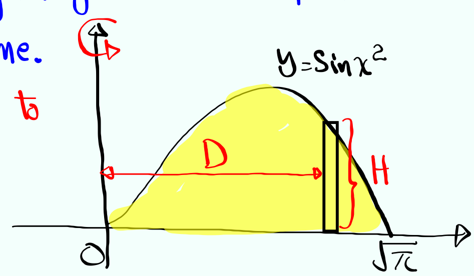
Rotate the region given below by Y-axis.

Find the Volume.

Ref. Rect. Parallel to A.O.R.

Shell

$$V=\int_a^b 2\pi D H\,dx$$




$D=x$ $\quad H=\sin x^2$

$$=\int_0^{\sqrt{\pi}} 2\pi\, x \sin x^2\,dx$$

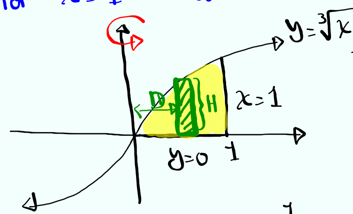
$u=x^2$

$du=2x\,dx$

$$=\pi\int_0^{\pi}\sin u\,du = -\cos u\Big|_0^{\pi} = -\pi\left[\overset{-1}{\cos\pi} - \overset{-1}{\cos 0}\right]$$

$$=-\pi(-2)=\boxed{2\pi}$$

Rotate the region bound by $y = \sqrt[3]{x}$, $y = 0$, and $x = 1$ about y-axis. Find the Volume.




Ref. Rect. parallel to A.O.R.

Shell

$D = x \quad H = \sqrt[3]{x}$

$$V = \int_0^1 2\pi D H \, dx$$

$$= \int_0^1 2\pi x \sqrt[3]{x} \, dx$$

$$= 2\pi \int_0^1 x^{4/3} \, dx$$

$$= 2\pi \left. \frac{x^{\frac{7}{3}}}{7/3} \right|_0^1$$

$$= \frac{6\pi}{7} \left. x^{7/3} \right|_0^1$$

$$= \frac{6\pi}{7} \left( 1^{7/3} - 0^{7/3} \right) = \boxed{\frac{6\pi}{7}}$$

$x \cdot \sqrt[3]{x} =$

$x \cdot x^{1/3} =$

$x^{\frac{4}{3}}$

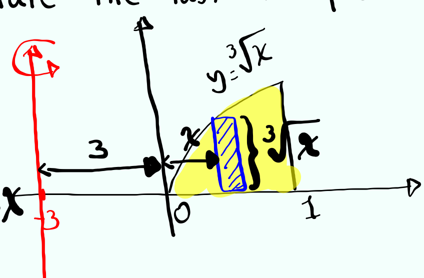
Jul 30-9:55 AM

$\boxed{\text{Set-up the integral}}$ to find the volume when we rotate the last example by

1) $x = -3$.

Shell

$$V = \int_0^1 2\pi \underbrace{(x+3)}_{D} \underbrace{\sqrt[3]{x}}_{H} \, dx$$

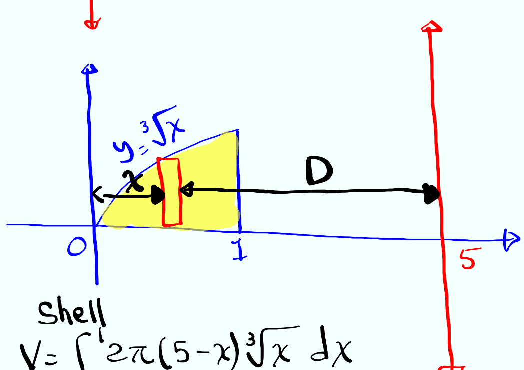



2) $x = 5$

$x + D = 5$

$D = 5 - x$

$H = \sqrt[3]{x}$

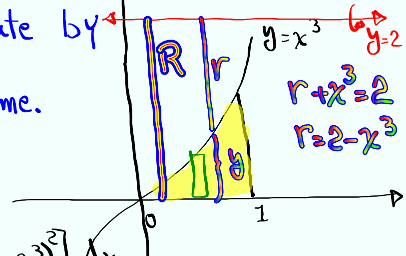



Shell

$$V = \int_0^1 2\pi (5 - x) \sqrt[3]{x} \, dx$$

Jul 30-10:01 AM

Consider the region bounded by $y=x^3$, $y=0$ (x-axis), and $x=1$. Rotate by




$r+x^3=2$

$r=2-x^3$

1) $y=2$, Find Volume.

washer

$R=2 \qquad r=2-x^3$

$$V=\int_0^1 \pi\left[2^2-(2-x^3)^2\right]dx$$

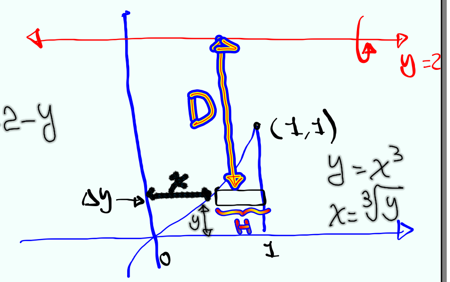
Shell

$y+D=2 \qquad D=2-y$

$x+H=1$

$H=1-x$

$H=1-\sqrt[3]{y}$




$y=x^3$

$x=\sqrt[3]{y}$

$$V=\int_0^1 2\pi(2-y)(1-\sqrt[3]{y})\,dy$$

Jul 30-10:09 AM

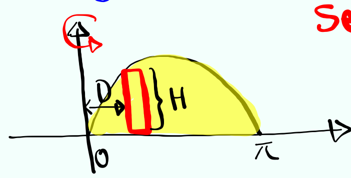
Rotate the region in QI bounded by $y=\sin x$, $y=0$, $0\le x\le\pi$ about

**Set-up integral for volume.**

1) y-axis

$D=x$

$H=\sin x$

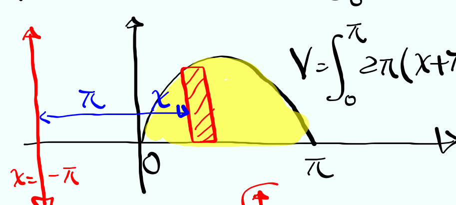



$$V=\int_0^{\pi} 2\pi x\sin x\,dx$$

2) $x=-\pi$

$D=x+\pi$

$H=\sin x$



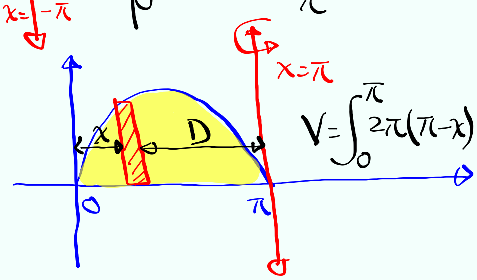

$$V=\int_0^{\pi} 2\pi(x+\pi)\sin x\,dx$$

3) $x=\pi$

$x+D=\pi$

$\boxed{D=\pi-x}$

$H=\sin x$




$$V=\int_0^{\pi} 2\pi(\pi-x)\sin x\,dx$$

Jul 30-10:19 AM

Fundamental theorem of Calculus

Suppose $f(x)$ is Cont. on $[a,b]$

If $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$

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If $g(x)=\int_{u(x)}^{v(x)} f(t)\,dt$, $g'(x)=f(v(x))\cdot v'(x)-f(u(x))\cdot u'(x)$

ex: $g(x)=\int_{\boxed{1}}^{\boxed{x^2}} \frac{1}{t^4+5}\,dt$

$$g'(x)=\frac{1}{(x^2)^4+5}\cdot 2x-\frac{1}{1^4+5}\cdot 0$$

$$=\boxed{\frac{2x}{x^8+5}}$$

Jul 30-10:30 AM

$$g(x)=\int_{\boxed{2x}}^{\boxed{3x}} \boxed{\frac{t^2-1}{t^2+1}}\,dt$$

$$g'(x)=\frac{(3x)^2-1}{(3x)^2+1}\cdot 3-\frac{(2x)^2-1}{(2x)^2+1}\cdot 2$$

$$=\frac{3(9x^2-1)}{9x^2+1}-\frac{2(4x^2-1)}{4x^2+1}$$

Jul 30-10:36 AM

$$g(x) = \int_{x^2}^{x^3} \cos t^2\,dt$$

1) $g(0) = \int_{0^2}^{0^3} \cos t^2\,dt = \int_0^0 \cos t^2\,dt = 0$

2) $g(1) = \int_{1^2}^{1^3} \cos t^2\,dt = \int_1^1 \cos t^2\,dt = 0$

3) $g'(x) = \cos(x^3)^2 \cdot 3x^2 - \cos(x^2)^2 \cdot 2x$

$= 3x^2 \cos x^6 - 2x \cos x^4$

$g'(1) = 3 \cdot 1^2 \cos 1^6 - 2 \cdot 1 \cos 1^4$

$= 3\cos 1 - 2\cos 1 = \cos 1 \approx .54$

Jul 30-10:39 AM

Discuss the concavity of

$$f(x) = \int_0^x \frac{t^2}{t^2+t+2}\,dt$$

$f'(x) = \frac{x^2}{x^2+x+2} \cdot 1 - \frac{0^2}{0^2+0+2} \cdot 0$

$f'(x) = \frac{x^2}{x^2+x+2}$

$f''(x) = \frac{2x(x^2+x+2) - x^2(2x+1)}{(x^2+x+2)^2}$

$= \frac{\cancel{2x^3} + 2x^2 + 4x - \cancel{2x^3} - x^2}{(x^2+x+2)^2} = \frac{x^2+4x}{(x^2+x+2)^2}$

P.I.P.

$f''(x) = 0 \quad x^2+4x=0 \quad x=0 \quad x=-4$

$f''(x)$ und. $\quad x^2+x+2=0$ No real Soln.

$(-\infty,-4)$ CU $(0,\infty)$ $\quad (-4,0)$ CD

| | | -4 | | 0 | |
|---|---|---|---|---|---|
| $f''(x)$ | + | | − | | + |
| | | I.P. | | I.P. | |

Jul 30-10:46 AM

$$f(x) = \int_0^x \frac{1}{1+t^2}\,dt + \int_0^{1/x} \frac{1}{1+t^2}\,dt \text{ for}$$

$(0, \infty)$

1) Find $f'(x)$

$$f'(x) = \frac{1}{1+x^2} \cdot 1 + \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2}$$

$$= \frac{1}{1+x^2} - \frac{1}{1+\frac{1}{x^2}} \cdot \frac{1}{x^2}$$

$$= \frac{1}{1+x^2} - \frac{1}{x^2+1} = \boxed{0}$$

2) Discuss $f(x)$

Since $f'(x) = 0$
then $f(x) = C$.

Jul 30-10:54 AM

Show $\int_{-1}^{1} \sqrt{1+x^2}\,dx \geq 2$

Hint $\sqrt{1+x^2} \geq \sqrt{1} \Rightarrow \sqrt{1+x^2} \geq 1$

$$\int_{-1}^{1} \sqrt{1+x^2}\,dx \geq \int_{-1}^{1} 1\,dx$$

$$\geq x\Big|_{-1}^{1} = (1-(-1))$$

$$= 2$$

$$\boxed{\int_{-1}^{1} \sqrt{1+x^2}\,dx \geq 2}$$

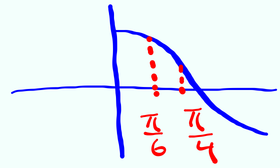
Jul 30-11:01 AM

Show $\frac{\sqrt{2}\,\pi}{24} \le \int_{\pi/6}^{\pi/4} \cos x \, dx \le \frac{\sqrt{3}\,\pi}{24}$

$\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

Hint: $\frac{\sqrt{2}}{2} \le \cos x \le \frac{\sqrt{3}}{2}$




$\int \qquad \int \qquad \int$

Jul 30-11:05 AM